CFT & gravity Week 5

Unitarity bounds

a) Unitarity implies that $\| P^2 |\mathcal{O}\rangle \|^2 \geq 0$, that is

$$\langle \mathcal{O} | K^2 P^2 | \mathcal{O} \rangle \geq 0$$

$$\langle \mathcal{O} | K^2 P^2 | \mathcal{O} \rangle = \langle \mathcal{O} | [K^2, P^2] | \mathcal{O} \rangle$$ ($K_\mu |\mathcal{O}\rangle = 0$, we choose $\mathcal{O}$ to be a primary)

$$[K^2, P^2] = K^\mu [K_\mu, P^2] + [K_\mu, P^2] K^\mu$$

zero on $|\mathcal{O}\rangle$

$$\rightarrow K^\mu P^\nu [K_\mu, P_\nu] + K^\mu [K_\mu, P_\nu] P^\nu$$

And we'll need:

$$[K_\mu, P_\nu] = 2\delta_{\mu\nu} D - 2i M_{\mu\nu}$$

$$[D, P_\nu] = P_\nu$$

$$[M_{\mu\nu}, P_\sigma] = i\left(\delta_{\mu\sigma} P_\nu - \delta_{\nu\sigma} P_\mu\right)$$

So: $$0 \leq \langle \mathcal{O} | K^\mu P^\nu \left(2\delta_{\mu\nu} D - 2i\cancel{M_{\mu\nu}}\right) | \mathcal{O} \rangle +$$ $M_{\mu\nu}|\mathcal{O}\rangle = 0$ (scalar)

$$+ \langle \mathcal{O} | K^\mu \left(2\delta_{\mu\nu} D - 2i M_{\mu\nu}\right) P^\nu | \mathcal{O} \rangle$$

$$= 2\Delta \langle \mathcal{O}| K^\mu P_\mu |\mathcal{O}\rangle +$$
$$+ 2\langle \mathcal{O}| K^\mu \left(\delta_{\mu\nu}[D, P^\nu] + \delta_{\mu\nu} P^\nu D - i[M_{\mu\nu}, P^\nu] \right)|\mathcal{O}\rangle$$

!

$$= 2\Delta \langle \mathcal{O}| K^\mu P_\mu |\mathcal{O}\rangle + 2\langle \mathcal{O}| K^\mu P_\mu |\mathcal{O}\rangle + 2\Delta \langle \mathcal{O}| K^\mu P_\mu |\mathcal{O}\rangle$$
$$+ 2\langle \mathcal{O}| K^\mu \left(\delta_\mu{}^\nu P_\nu - \delta_\nu^\nu P_\mu \right)|\mathcal{O}\rangle$$
$$= 2\left(2\Delta + 1 + (1-d)\right) \langle \mathcal{O}| K^\mu P_\mu |\mathcal{O}\rangle$$
$$= 4\left(\Delta + \frac{2-d}{2}\right) \sum_\mu \| P_\mu |\mathcal{O}\rangle \|^2$$

Since unitarity also implies $\| P_\mu |\mathcal{O}\rangle \|^2 \geq 0$,

$$\boxed{\Delta \geq \frac{d-2}{2}}$$

The inequality is saturated when $\| P^2 |\mathcal{O}\rangle \|^2 = 0$.

The only vector with zero length is the null vector, so

$$P^2 |\mathcal{O}\rangle = 0$$

Now:
$$P^2|\mathcal{O}\rangle = P^2 \mathcal{O}(0)|0\rangle$$
$$= [P^2, \mathcal{O}(0)]|0\rangle \quad \text{(invariance of vacuum)}$$
$$= -\Box \mathcal{O}(0)|0\rangle$$

$\Box \mathcal{O}(x) = 0$ in all correlation functions $\rightsquigarrow$ $\mathcal{O}(x)$ is a free field ∎

b) The norm $\langle \mathcal{O}^\mu | K_\mu P_\nu | \mathcal{O}^\nu \rangle$

$$= \langle \mathcal{O}^\mu | [K_\mu, P_\nu] | \mathcal{O}^\nu \rangle = 2 \langle \mathcal{O}^\mu | \delta_{\mu\nu} D - i M_{\mu\nu} | \mathcal{O}^\nu \rangle$$

$$= 2\Delta \langle \mathcal{O}^\mu | \mathcal{O}_\mu \rangle - 2i \langle \mathcal{O}^\mu | (M_{\mu\nu})^\nu{}_\beta | \mathcal{O}^\beta \rangle$$

$$(M_{\mu\nu})^\nu{}_\beta = i \left(\delta_{\nu\beta} \delta_\mu{}^\nu - \delta_{\mu\beta} \delta_\nu^\nu \right) = i(1-d)\, \delta_{\mu\beta}$$

$$= 2(\Delta + 1 - d) \sum_\mu \| | \mathcal{O}_\mu \rangle \|^2 \underset{\text{unitarity}}{\geq} 0$$

$$\Longrightarrow \quad \Delta \geq d-1$$

Equality when:

$$P_\mu | \mathcal{O}^\mu \rangle = 0$$

same reasoning as in a): $\longrightarrow \partial_\mu \mathcal{O}^\mu(x) = 0$

$\mathcal{O}^\mu$ is a conserved current.

c) See Mathematica code for a proof that C is a Casimir.

$$C | \mathcal{O} \rangle = \left(\Delta^2 - \frac{1}{2} [K_\mu, P^\mu] \right) | \mathcal{O} \rangle =$$

$$= (\Delta^2 - d\Delta) | \mathcal{O} \rangle = \Delta(\Delta - d) | \mathcal{O} \rangle$$

$$C|\mathcal{O}^\lambda\rangle = \left(\Delta^2 - d\Delta + \frac{1}{2} M_{\mu\nu} M^{\mu\nu}\right)|\mathcal{O}^\lambda\rangle$$

$\frac{1}{2} M_{\mu\nu} M^{\mu\nu}$ is the Casimir of the rotation group, aka, L^2.
But let's compute directly:

$$(M_{\mu\nu})^\lambda{}_\beta \, (M^{\mu\nu})^\gamma{}_\alpha = -\left(\delta_{\nu\beta}\delta_\mu^\lambda - \delta_{\mu\beta}\delta_\nu^\lambda\right)\left(\delta^\nu{}_\alpha \delta^{\mu\gamma} - \delta^\mu_\alpha \delta^{\nu\gamma}\right)$$
$$= -2\left(\delta_{\alpha\beta}\,\delta^{\lambda\gamma} - \delta_\beta{}^\gamma \delta^\lambda_\alpha\right) \qquad (1)$$

So in particular

$$\frac{1}{2} M_{\mu\nu} M^{\mu\nu}|\mathcal{O}^\lambda\rangle = \frac{1}{2}(M_{\mu\nu})^\lambda{}_\beta \, (M^{\mu\nu})^\beta{}_\alpha |\mathcal{O}^\alpha\rangle =$$
$$= (d-1)\;|\mathcal{O}^\lambda\rangle$$

(In $d=3$ this is the usual $j(j+1)\big|_{j=1} = 2$).

So:
$$C|\mathcal{O}^\lambda\rangle = \left(\Delta(\Delta-d) + (d-1)\right)|\mathcal{O}^\lambda\rangle$$

* * *

(<u>OBSERVATION</u>: the Casimir has the same eigenvalue on all members of a conformal family. For instance, if $\mathcal{O}$ is a scalar primary,

e.g. $$C\,P_\mu|\mathcal{O}\rangle = P_\mu\, C|\mathcal{O}\rangle = \Delta(\Delta-d)\,P_\mu|\mathcal{O}\rangle.$$)

* * *

d) Compute the norm:

$$0 \leq \langle \mathcal{O}^{\mu_1 \cdots \mu_\ell} | K_{\mu_1} P_{\nu_1} | \mathcal{O}^{\nu_1}{}_{\mu_2 \cdots \mu_\ell} \rangle$$

$$= 2\Delta \langle \mathcal{O}^{\mu_1 \cdots \mu_\ell} | \mathcal{O}_{\mu_1 \cdots \mu_\ell} \rangle - 2i \langle \mathcal{O}^{\mu_1 \cdots \mu_\ell} | M_{\mu_1 \nu_1} | \mathcal{O}^{\nu_1}{}_{\mu_2 \cdots \mu_\ell} \rangle$$

$$M_{\mu_1 \nu_1} | \mathcal{O}^{\nu_1 \mu_2 \cdots \mu_\ell} \rangle = i(1-d) | \mathcal{O}_{\mu_1}{}^{\mu_2 \cdots \mu_\ell} \rangle + \sum_{J=2}^{\ell} (M_{\mu_1 \nu_1})^{\mu_J}{}_{\beta} | \mathcal{O}^{\nu_1 \cdots \beta \cdots \mu_\ell} \rangle$$

$$(M_{\mu_1 \nu_1})^{\mu_J}{}_{\beta} = i \left(\delta_{\nu_1 \beta} \delta_{\mu_1}{}^{\mu_J} - \delta_{\mu_1 \beta} \delta_{\nu_1}{}^{\mu_J} \right)$$

By tracelessness:

$$= i(1-d) | \mathcal{O}_{\mu_1}{}^{\mu_2 \cdots \mu_\ell} \rangle - i \sum_{J=2}^{\ell} | \mathcal{O}^{\mu_J \mu_2 \cdots}{}_{\mu_1}{}^{\cdots \mu_\ell} \rangle =$$

$$= -i(d + \ell - 2) | \mathcal{O}_{\mu_1}{}^{\mu_2 \cdots \mu_\ell} \rangle$$

So: $$0 \leq \left(2\Delta - 2(d + \ell - 2) \right) \sum_{\{\mu\}} \| | \mathcal{O}_{\mu_1 \cdots \mu_\ell} \rangle \|^2$$

$$\Rightarrow \quad \Delta \geq d - 2 + \ell \, .$$

In computing the action of C, we only need to change the eigenvalue of $\frac{1}{2} M_{\mu\nu} M^{\mu\nu}$.

$$\frac{1}{2} M_{\mu\nu} M^{\mu\nu} |\mathcal{O}^{\lambda_1 \dots \lambda_\ell}\rangle =$$

$$= \frac{1}{2} \sum_i (M^{\mu\nu})^{\lambda_i}{}_{\alpha} \, M_{\mu\nu} |\mathcal{O}^{\lambda_1 \dots \alpha \dots \lambda_\ell}\rangle =$$

$$= \frac{1}{2} \sum_i (M^{\mu\nu})^{\lambda_i}{}_{\alpha} \sum_{\substack{J \\ (J \neq i)}} (M_{\mu\nu})^{\lambda_J}{}_{\beta} |\mathcal{O}^{\lambda_1 \dots \alpha \dots \beta \dots \lambda_\ell}\rangle +$$

$$+ \frac{1}{2} \sum_i (M^{\mu\nu})^{\lambda_i}{}_{\alpha} (M^{\mu\nu})^{\alpha}{}_{\beta} |\mathcal{O}^{\lambda_1 \dots \beta \dots \lambda_\ell}\rangle$$

Recall (1):

$$(M^{\mu\nu})^{\lambda_i}{}_{\alpha} (M_{\mu\nu})^{\lambda_J}{}_{\beta} = -2\left(\delta_{\alpha\beta}\, \delta^{\lambda_i \lambda_J} - \delta_{\alpha}{}^{\lambda_J} \delta_{\beta}{}^{\lambda_i}\right)$$

$$\ell(\ell-1) |\mathcal{O}^{\lambda_1 \dots \lambda_\ell}\rangle + (d-1)\ell |\mathcal{O}^{\lambda_1 \dots \lambda_\ell}\rangle$$

$$\hookrightarrow \quad \frac{1}{2} M_{\mu\nu} M^{\mu\nu} |\mathcal{O}^{\lambda_1 \dots \lambda_\ell}\rangle = \ell(\ell + d - 2) |\mathcal{O}^{\lambda_1 \dots \lambda_\ell}\rangle$$

this generalizes $\ell(\ell+1)$ in $d=3$

So we get that on a spin ℓ primary:

$$C = \Delta(\Delta - d) + \ell(\ell + d - 2)$$

$\boxed{\text{Analytic structure of the two-point function}}$

a) $\langle 0|\hat{\mathcal{O}}(\tau_1, \vec{x}_1)\hat{\mathcal{O}}(\tau_2, x_2)|0\rangle =$

$$= \langle 0|\hat{\mathcal{O}}(0,\vec{x}_1)\, e^{\hat{H}(\tau_2-\tau_1)}\,\hat{\mathcal{O}}(0,\vec{x}_2)|0\rangle =$$

$$\int dE \int d\vec{k}\, \langle 0|\hat{\mathcal{O}}(0,\vec{x}_1)\,.\, e^{\hat{H}(\tau_2-\tau_1)}|E,\vec{k}\rangle\langle\vec{k},E|\hat{\mathcal{O}}(0,\vec{x}_2)|0\rangle$$

$$= \int_{E_{min}}^{\infty} dE \int d\vec{k}\, \langle 0|\hat{\mathcal{O}}(0,\vec{x}_1)|E,\vec{k}\rangle\langle\vec{k},E|\hat{\mathcal{O}}(0,\vec{x}_2)|0\rangle\, e^{E(\tau_2-\tau_1)}$$

Unless $\langle E,\vec{k}|\hat{\mathcal{O}}|0\rangle$ is suppressed more than exponentially as $E\to\infty$, the result is divergent. But we know the spectral density from exercise 2.7.1:

$$= \int_{E_{min}}^{\infty} dE \int d\vec{k}\, \underbrace{|\langle\vec{k},E|\hat{\mathcal{O}}(0,0)|0\rangle|^2}_{\rho(\vec{k},E)}\, e^{i(\vec{x}_1-\vec{x}_2)\cdot\vec{k}}\, e^{E(\tau_2-\tau_1)}$$

And since $\rho(\vec{k},E) \sim (E^2-k^2)^{\Delta-\frac{d}{2}}$ (already from dimensional analysis)

the exponential factor wins and the result is infinite when $\tau_2 > \tau_1$.

$$F(\tau) = \frac{1}{(\vec{x}^2 + \tau^2)^{\Delta}}$$

b) $$F(\tau) = \left((\tau + i|\vec{x}|)(\tau - i|\vec{x}|) \right)^{-\Delta}$$

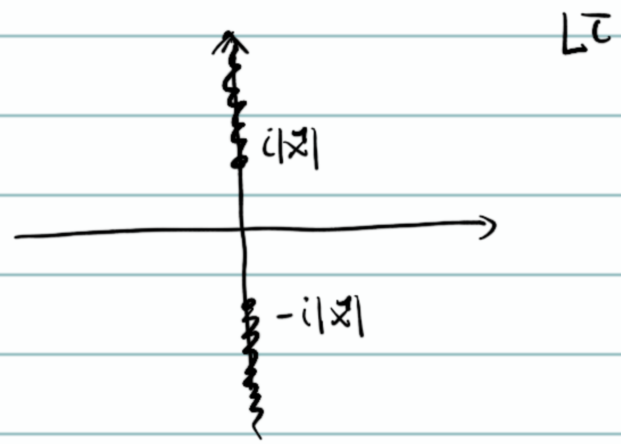
Two branch points sit at $\tau = \pm i|x|$

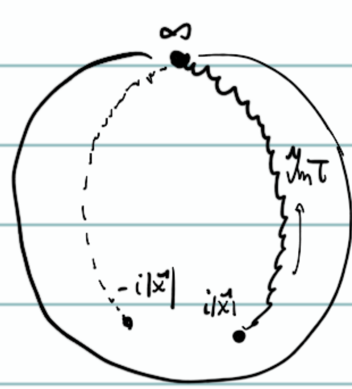
Is $\tau = \infty$ also a branch point?

$$F\left(\frac{1}{z}\right) = \left[z^{-2} (1 + iz|\vec{x}|)(1 - iz|\vec{x}|) \right]^{-\Delta}$$

There is a non trivial monodromy around $z = 0$, so also $\tau = \infty$ is a branch point.

Figure 1 :





c) The correlator $F(\tau)$ is time ordered in Euclidean time.

With

$$\tau = it + \varepsilon$$

we establish the ordering via a small positive or negative ε.

[COMMENT: people often use the variable $x^0 = -i\tau = t - i\varepsilon$ so that $\mathrm{Re}\, x^0 =$ Lorentzian time. Then:

"The rightmost operator is the one with the largest $\mathrm{Im}\, x^0$"]

$$\langle 0| [\mathcal{O}(it,\vec{x}), \mathcal{O}(0)] |0\rangle = \langle 0| \mathcal{O}(it,\vec{x})\, \mathcal{O}(0) |0\rangle$$

$$- \langle 0| \mathcal{O}(0)\, \mathcal{O}(it,\vec{x}) |0\rangle =$$

$$= \lim_{\varepsilon \to 0} \Big(\langle 0| \mathcal{O}(it+\varepsilon, \vec{x})\, \mathcal{O}(0) |0\rangle - \langle 0| \mathcal{O}(0)\, \mathcal{O}(it-\varepsilon, \vec{x}) |0\rangle \Big)$$

The expectation values are now Euclidean time ordered, so we can replace them by correlators:

$$= \lim_{\varepsilon \to 0} \Big(F(it+\varepsilon) - F(it-\varepsilon) \Big) =$$

$$= \mathrm{disc}_{\tau} F(\tau)$$

so the discontinuity computes the correlator.

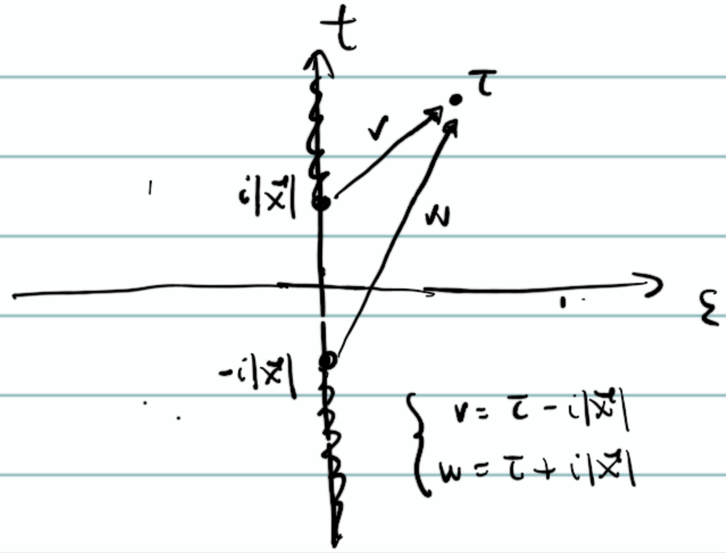
Comparing with Figure 1,

across the cuts there is a non-zero discontinuity,

while if $|t| < |\vec{x}| \longrightarrow \langle 0| [\mathcal{O}(it,\vec{x}), \mathcal{O}(0)] |0\rangle = 0$

This is compatible w/ causality that implies that the commutator vanishes for space-like separated points.

Let's compute the discontinuity:



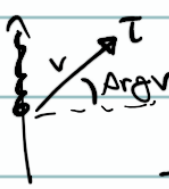
$$\begin{cases} v = \tau - i|\vec{x}| \\ w = \tau + i|\vec{x}| \end{cases}$$

$$F(\tau) = \left[(\tau + i|\vec{x}|)(\tau - i|\vec{x}|)\right]^{-\Delta}$$

$$= (w\,v)^{-\Delta}$$

We must choose phases so that $F(\tau) > 0$ when $\tau \in \mathbb{R}$

We can choose $\operatorname{Arg} v \in \left(-\frac{3}{2}\pi, \frac{1}{2}\pi\right)$, $\operatorname{Arg} w \in \left(-\frac{1}{2}\pi, \frac{3}{2}\pi\right)$,

that is 

and similarly for w, so that $\operatorname{Arg} v = -\operatorname{Arg} w$ when $\tau \in \mathbb{R}$, Then:

o $t > |\vec{x}|$ $\quad w = |w| e^{i\frac{\pi}{2}}$ $\quad e^{-i\pi\Delta} - e^{i\pi\Delta}$

$$\operatorname{disc} F = \left(|w|e^{i\frac{\pi}{2}}|v|e^{+i\frac{\pi}{2}}\right)^{-\Delta} - \left(|w|e^{i\frac{\pi}{2}}|v|e^{-i\frac{3}{2}\pi}\right)^{-\Delta} =$$
$$= |vw|^{-\Delta}\left(-2i\sin\pi\Delta\right) = |t^2 - \vec{x}^2|^{-\Delta}\left(-2i\sin\pi\Delta\right)$$

$\odot\ t < -|\vec{x}|$

$$\text{disc}\, F = \left(|w|\, e^{-i\frac{\pi}{2}} |v|\, e^{-i\frac{\pi}{2}}\right)^{-\Delta} - \left(|w|\, e^{i\frac{3}{2}\pi} |v|\, e^{-i\frac{\pi}{2}}\right)^{-\Delta} =$$

$$= |t^2 - \vec{x}^2|^{-\Delta} \left(2i \sin \pi\Delta\right)$$

All together:

$$\langle 0 | [\mathcal{O}(it, \vec{x}), \mathcal{O}(0)] | 0 \rangle = -2i \sin \pi\Delta \left(\theta(x^0 - |\vec{x}|) - \theta(-x^0 - |\vec{x}|)\right)$$

$$= -2i \sin \pi\Delta\, |t^2 - \vec{x}^2|^{-\Delta} \left(\theta(t - |\vec{x}|) - \theta(-t - |\vec{x}|)\right) \blacksquare$$

Notice that the minus sign between the θ functions is required by translational invariance:

$$\langle 0 | [\mathcal{O}(-it, -\vec{x}), \mathcal{O}(0)] | 0 \rangle \overset{\text{translational invariance}}{=} \langle 0 | [\mathcal{O}(0), \mathcal{O}(it, \vec{x})] | 0 \rangle =$$

$$= -\langle 0 | [\mathcal{O}(it, \vec{x}), \mathcal{O}(0)] | 0 \rangle$$

So sending $t \to -t$ must reverse the sign of the commutator.

Matching on the lattice

a) Spacetime symmetries

- $O(2) \longrightarrow D_4 = \{a, b \,/\, a^4 = 1,\ b^2 = 1,\ (ab)^2 = 1\}$

(Rotation + parity) is broken to D_4, symmetry group of a square:

$a \equiv$ rotation by $\frac{\pi}{2}$

$b \equiv$ reflection wrt a horizontal (or vertical) line.

- translations $\longrightarrow$ discrete translations (lattice spacing)

- scale + special conformal $\longrightarrow$ broken by the lattice, but recovered at large distances (see point d)

<u>Internal</u>: $\mathbb{Z}_2 \longrightarrow$ preserved on the lattice

b) Operators which contribute to S_{ij} transform as follows:

- Scalars under D_4 (<u>NOT</u> $O(2)$)
- $\mathbb{Z}_2$ odd

Need to see which representations under $O(2)$ contain scalars under D_4.

It is convenient for notation to define complex coordinates:

$$z = x + iy$$
$$\bar{z} = x - iy$$

IRREPS of $O(2)$ are two dimensional and labeled by an integer spin $J \in \mathbb{N}$:

$$\left\{ O^{J}(z,\bar{z}),\ O^{-J}(z,\bar{z}) \right\}$$

rotations: $\mathcal{O}^j(e^{i\varphi}z, e^{-i\varphi}\bar{z}) = e^{i\varphi j}(\mathcal{O}^j)'(z,\bar{z})$

parity: $\mathcal{O}^j(\bar{z}, z) = (\mathcal{O}^{-j})'(z,\bar{z})$
$(y \to -y)$

(This is nothing fancy: you can think of $\mathcal{O}$ as a traceless symmetric tensor w/ j indices. Since in complex coordinates the metric is $ds^2 = dz d\bar{z}$, the only non zero components are $\mathcal{O}^j \equiv \mathcal{O}_{\underbrace{\bar{z}\cdots\bar{z}}_{j}}$ and $\mathcal{O}^{-j} \equiv \mathcal{O}_{\underbrace{z\cdots z}_{j}}$

and they are exchanged by parity)

Now: if $j = 4n \quad n \in \mathbb{N}$

- Rotations $\in D_4$ leave the operator invariant
- The combination $\mathcal{O}^j + \mathcal{O}^{-j}$ is also invariant under parity.

Bottom line: we look for operators w/ spin

$j = 4n$

$$
\begin{aligned}
\Big[\, S = \; & c_1\, a^{\Delta_\sigma}\, \sigma \qquad\qquad \Delta_\sigma = \tfrac{1}{8} \\
& + c_2\, a^{\Delta_\sigma + 2}\, \partial_z \partial_{\bar z} \sigma \\
& + c_3\, a^{\Delta_\sigma + 4}\, \partial_z^2 \partial_{\bar z}^2 \sigma \\
& + c_4\, a^{\Delta_\sigma + 4} \left(\partial_z^4 \sigma + \partial_{\bar z}^4 \sigma \right) \\
& + c_5\, a^{\Delta_\sigma + 4} \left(\partial_z \Theta^{j=-3} + \partial_{\bar z} \Theta^{j=3} \right) \Big] \\
& + \cdots
\end{aligned}
$$

c) • The 2 point function of descendants is just found taking derivatives of the two-point function of σ (a scalar)

$$\langle \sigma(z,\bar{z})\, \sigma(0,0) \rangle = \frac{1}{(z\bar{z})^{\Delta_\sigma}}$$

• The two-point function of a primary of spin J is found as follows:

• scaling: $\langle \mathcal{O}^J(z,\bar{z})\, \mathcal{O}^J(0,0) \rangle = \frac{f(z/\bar{z})}{(z\bar{z})^{\Delta}}$

• rotations: $f(z/\bar{z}) = \left(\frac{z}{\bar{z}}\right)^J$

$\Rightarrow$ $\langle \mathcal{O}^J(z,\bar{z})\, \mathcal{O}^J(0,0) \rangle = \frac{1}{z^{\Delta-J}\, \bar{z}^{\Delta+J}}$

Customary notation in 2d: $h \equiv \frac{\Delta-J}{2}$ $\bar{h} = \frac{\Delta+J}{2}$

"holomorphic" and "antiholomorphic" dimensions

Note what about a two-point function of different primaries w/ spin?
Like we saw in higher d for scalars, they vanish, but this requires special conformal transformations.

- A way to see this in general d is as follows:
  1. Map one op. to zero and the other to ∞ via inversion, you obtain the matrix element:

$$\langle \Delta, J | \Delta', J' \rangle$$

  2. Then act w/ D:

$$\langle \Delta, J | D | \Delta', J' \rangle = \Delta' \langle \Delta, J | \Delta', J' \rangle$$
$$= \Delta \langle \Delta, J | \Delta', J' \rangle$$

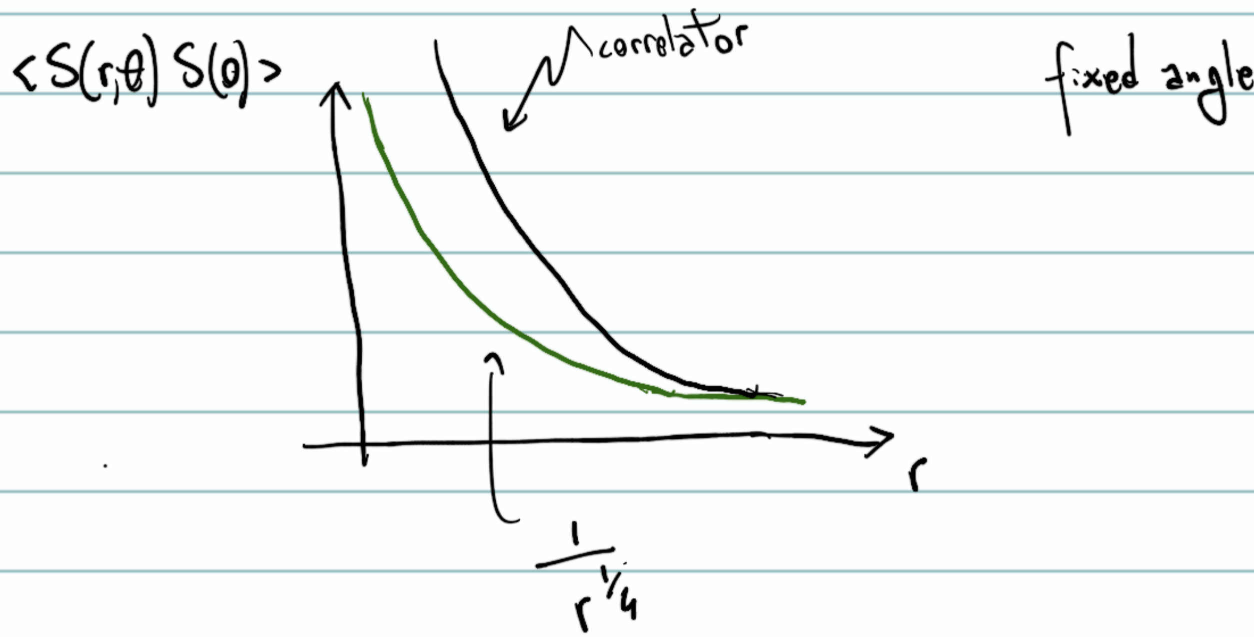
$\Rightarrow$ $\Delta = \Delta'$ or matrix element vanishes

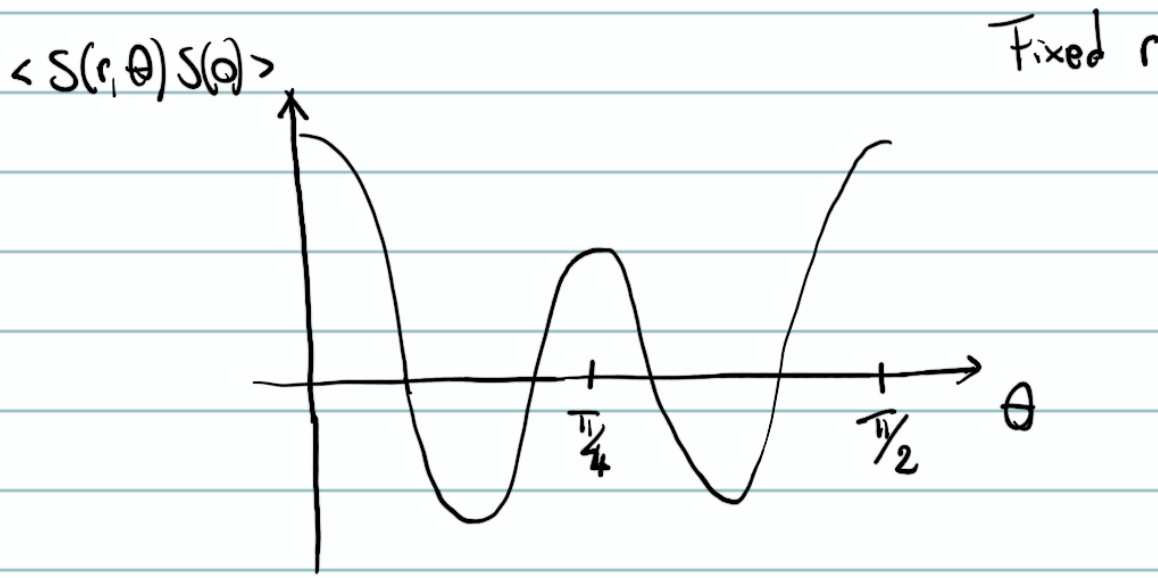
  3. Act w/ Casimir of rotation group (same as before)
$\Rightarrow$ $J = J'$ or matrix element vanishes

- You can prove this directly in 2d using holomorphic factorization.

From here it is a matter of taking derivatives, which you should do in Mathematica.

If you plot the result (see notebook), you'll see something like this:





There is a symmetry for $\theta \to \theta + \frac{\pi}{2}$ as expected.

but full rotational symmetry is only recovered as $r \to \infty$.

Play w/ the parameters in Mathematica to get a feeling for the size of the effect.

d) Each 2-point function contributing to $\langle SS \rangle$ is modified as follows:

$$\langle \mathcal{O}(x)\, \mathcal{O}'(0)\, e^{g\int d^2y\, a^{\Delta_{\varepsilon'}-2}\, \varepsilon'(y)} \rangle \qquad x = (r,\theta)$$

$$\simeq \langle \mathcal{O}(x)\, \mathcal{O}'(0) \rangle \qquad g\, a^{\Delta_{\varepsilon'}-2} \int d^2y\, \langle \mathcal{O}(x)\, \mathcal{O}'(0)\, \varepsilon'(y) \rangle$$

If we are naive, we get the right result by dimensional analysis:

$$\int d^2y\, \langle \mathcal{O}(x)\, \mathcal{O}'(0)\, \varepsilon'(y) \rangle \equiv F(x)$$

$F(x)$ has the same properties under rotation as $\langle \mathcal{O}(x)\, \mathcal{O}'(0) \rangle$ and has overall dimension $\Delta_{\mathcal{O}} + \Delta_{\mathcal{O}'} + \Delta_{\varepsilon'} - 2$

Therefore: $$F(x) = \frac{g(\theta)}{r^{\Delta_{\mathcal{O}} + \Delta_{\mathcal{O}'} + \Delta_{\varepsilon'} - 2}}$$

for some $g(\theta)$ which depends on the spin of $\mathcal{O}$ and $\mathcal{O}'$.

These are the additional powers generated at leading order in g.

Notice that, even if $\langle \mathcal{O}\mathcal{O}' \rangle = 0$, $\langle \mathcal{O}\mathcal{O}'\varepsilon' \rangle$ may be non-zero..
For instance, we could have a power

$$\sim g \frac{g(\theta)}{r^{\Delta_\sigma + \Delta_{\mathcal{O}3} + 1 + \Delta_{\varepsilon'} - 2}}$$

coming from $\langle \sigma\, \partial_z \mathcal{O}^{J=3} \varepsilon' \rangle$ even if $\langle \sigma\, \mathcal{O}^{J=3} \rangle = 0$

(If you know 2d CFT, you may go ahead and see if this really happens ...)

* The argument before is too naive, though.

Indeed, the integral

$$\int d^2y \; \langle \mathcal{O}(x)\, \mathcal{O}'(0)\, \varepsilon'(y) \rangle \;=\; \infty \quad !$$

- Indeed, as $y \to 0$:

$$\langle \mathcal{O}(x)\, \mathcal{O}'(0)\, \varepsilon'(y) \rangle \sim y^{-\Delta_{\varepsilon'} - \Delta_{\mathcal{O}'} + \Delta_{\mathcal{O}}}$$

Since $\Delta_{\varepsilon'} > 2$ (irrelevant operator)

convergence implies $\Delta_{\mathcal{O}} - \Delta_{\mathcal{O}'} > \Delta_{\varepsilon'} - 2 > 0$

- But if so, at the opposite singularity:

$$y \to x \qquad \sim \quad |x-y|^{-\Delta_{\varepsilon'} - \Delta_{\mathcal{O}} + \Delta_{\mathcal{O}'}}$$

diverges

This is not a tragedy, because of the lattice spacing a.

The integrals are cut-off when operators get to distances of order a.

But this does complicate the dimensional analysis, because a is a new scale:

if we cut off w/ spheres of radius a,

$$a^{\Delta_{\varepsilon'}-2}\int_{\substack{|y|>a \\ |y-x|>a}} d^2y \; \langle \mathcal{O}(x)\,\mathcal{O}'(0)\,\varepsilon'(y)\rangle = \frac{g(\theta)}{|x|^{\Delta_{\mathcal{O}}+\Delta_{\mathcal{O}'}}}\left(\frac{a}{|x|}\right)^{\Delta_{\varepsilon'}-2} f\!\left(\frac{a}{|x|}\right)$$

We are interested in the large $|x|$ limit, where conformal invariance is restored. As expected, this is the same as $a \to 0$

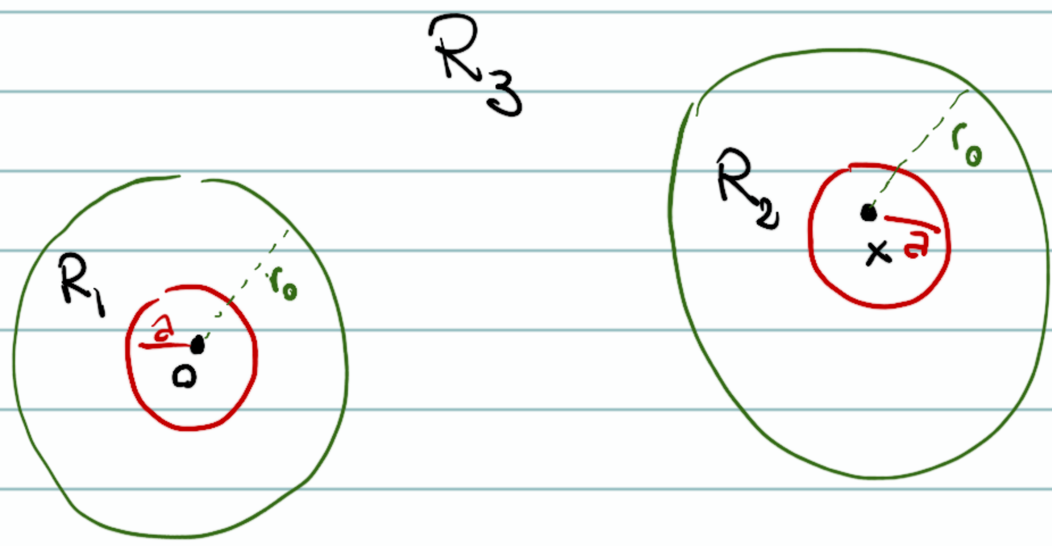
$f\left(\frac{a}{|x|}\right)$ has a power expansion in the argument, dictated by the short distance singularities:

$$f\left(\frac{a}{|x|}\right) \sim c_1 \left(\frac{a}{|x|}\right)^{-\Delta_{\varepsilon'} - \Delta_{\mathcal{O}'} + \Delta_{\mathcal{O}} + 2} \left(1 + \# \frac{a}{|x|} + \dots\right)$$

$$+ c_2 \left(\frac{a}{|x|}\right)^{-\Delta_{\varepsilon'} - \Delta_{\mathcal{O}} + \Delta_{\mathcal{O}'} + 2} \left(1 + \# \frac{a}{|x|} + \dots\right)$$

$$+ c_3$$

This can be seen as follows. Cut the integration domain in three regions R_1, R_2, R_3



The radius r_0 is arbitrary, and the result does not depend on it.

The integral over R_3 is independent of a.

It's a function of $\frac{|x|}{r_0}$, but the dependence on r_0 will eventually cancel.

Inside R_1 we can Taylor expand the integrand in powers of y:

$$\int_a^{r_0} d|y||y| \int_0^{2\pi} d\varphi \, \langle \mathcal{O}(x) \, \mathcal{O}'(0) \, \mathcal{E}'(y) \rangle$$

$$= \int_0^{2\pi} d\varphi \int_a^{r_0} dy \, |y| \sum_{n=0}^{\infty} c_n(\varphi, x) \, |y|^{-\Delta_{\mathcal{E}'} - \Delta_{\mathcal{O}'} + \Delta_{\mathcal{O}} + n}$$

$y = (|y|, \varphi)$

The c_n only depend on φ, not on $|y|$. Their form is not important

$$\sim k_1(x) \left[|a|^{-\Delta_{\mathcal{E}'} - \Delta_{\mathcal{O}'} + \Delta_{\mathcal{O}} + 2} - |r_0|^{-\Delta_{\mathcal{E}'} - \Delta_{\mathcal{O}'} + \Delta_{\mathcal{O}} + 2} \right]$$

$+ \ldots$ The dependence on a in the function f comes from here

The same happens in the region R_2.

In the end, we get the two series in a, and by dim. analysis there are series in $a/|x|$.

Dimensional analysis allows a function $c\left(\frac{|x|}{r_0}\right)$ to survive from region R_3 and the a indip. parts of the integrations in R_1 and R_2.

But the integral must be r_0 independent, so $c\left(\frac{|x|}{r_0}\right) = C_3$ the constant above.

The constant C_3 leads to the additional power found w/ the naive method.

The other powers? The full function becomes:

$$a^{\Delta_{\varepsilon'}-2} \int_{\substack{|y|>a \\ |y-x|>a}} d^2y \; \langle \mathcal{O}(x)\, \mathcal{O}'(0)\, \varepsilon'(y) \rangle = \frac{c_3\, g(\theta)}{|x|^{\Delta_\mathcal{O}+\Delta_{\mathcal{O}'}}} \left(\frac{a}{|x|}\right)^{\Delta_{\varepsilon'}-2}$$

$$+ \sum_{n=0}^{\infty} \frac{d_n\, g(\theta)}{|x|^{2\Delta_\mathcal{O}+n}}\, a^{\Delta_\mathcal{O}-\Delta_{\mathcal{O}'}}$$

$$+ \sum_{n=0}^{\infty} \frac{e_n\, g(\theta)}{|x|^{2\Delta_{\mathcal{O}'}+n}}\, a^{\Delta_{\mathcal{O}'}-\Delta_\mathcal{O}}$$

All these powers were already in $\langle S\, S \rangle$, they came from correlators of derivatives of $\mathcal{O}$ and of $\mathcal{O}'$, so their coefficient is just renormalized.

In fact you know of this phenomenon from QFT: this is just operator mixing. In the renormalized theory:

$$[\mathcal{O}]_r = c_0\, \mathcal{O} + c_1\, a^{\Delta_{\mathcal{O}'}-\Delta_\mathcal{O}}\, \mathcal{O}' + c_2\, a^{\Delta_{\mathcal{O}'}+1-\Delta_\mathcal{O}}\, \partial \mathcal{O}' + \dots$$

(This is slightly schematic because the precise mixing depends on the spin of $\mathcal{O}$ and $\mathcal{O}'$)